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Procedia Engineering 38 (2012) 3318 – 3330

Procedia Engineering

www.elsevier.com/locate/procedia

International Conference on Modelling, Optimisation and Computing (ICMOC 2012), April 10 - 11, 2012

Effect of damage parameters on vibration signatures of a cantilever beam

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Abstract

This paper addresses the fault detection of Multi cracked slender Euler–Bernoulli beams through the knowledge of changes in the natural frequencies and their measurements. The method is based on the approach of modelling a crack by rotational spring. The spring model of crack is applied to establish the frequency equation based on the dynamic stiffness of multiple cracked beams. Theoretical expressions for beams by natural frequencies have been formulated to find out the effect of crack depths on natural frequencies and mode shapes. The equation is the basic instrument in solving the multi-crack detection of beam. Results obtained for a cantilever beam with two cracks analysis show an efficient state of the research on multiple cracks effects and their identification.

Keywords: Damage detection; Vibration analysis; Cantilever beam; Mode shape; Compliance matrix

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doi:10.1016/j.proeng.2012.06.384

1. Introduction

The problem of damage and crack detection in structures or in machine components has acquired important role in last two decades. However, the studies are mainly dealt with single crack. The research in the past few decades on cracked structures and rotors is well documented in a review paper by Dimarogona [1] and more recently by Parhi et al. [2]. If the structure is cracked in at least two positions, the problem of crack sizing and location becomes decidedly more complex.

Nomenclature

a_1	depth of first crack
a_2	depth of second crack
A	cross-sectional area of the beam
A_i i = 1to 18	unknown coefficients of matrix A
B	width of the beam
E	Young's modulus of elasticity of the beam material
J	strain-energy release rate
$K_{I,i}$ i = 1,2	Stress intensity factors for P_i loads
K_{ij}	local flexibility matrix elements
L	length of the beam
L_1	location (length) of the first crack from fixed end
L_2	location (length) of the second crack from fixed end
P_i i=1,2	axial force (i=1), bending moment (i=2)
Q	stiff-ness matrix for free vibration.
Rcd	Relative crack depth
Rcl	Relative crack location
W	depth of the beam
β_1	relative first crack location $=L_1/L$
β_2	relative second crack location$=L_2/L$
μ	$A\rho$
ρ	mass-density of the beam
ξ_1	relative first crack depth = a_1/W
ξ_2	relative second crack depth = a_2/W

The effects of vibration response in beams having multiple cracks have of late drawn attention from researchers [3, 4]. Vibration-based crack detection is generally based on a change of transverse natural frequency of a component [5, 6]. In the modelling, a crack is represented by a rotational spring. The stiffness of the spring can be determined if the stress intensity factor (SIF) under a bending load is available. Chasalevris and Papadopoulos [7] have studied the dynamic behaviour of a cracked beam with two transverse surface cracks each being characterised by its depth, position and relative angle. A local compliance matrix of two degrees of freedom is calculated based on the stress intensity factors and. Patil and Maiti [8, 9] and later Baris [10] have utilized a method for prediction of location and size of multiple cracks based on measurement of natural frequencies for slender cantilever beams with two and three normal edge cracks. The damage index is an indicator of the extent of strain energy stored in the rotational spring. The same has been verified experimentally. Yoona Han-Ik et al. [11] have investigated the influence of two open cracks on the dynamic behavior of a double cracked simply supported beam both analytically and experimentally. The equation of motion is derived by using the Hamilton's principle and analyzed by numerical method. The simply supported beam is modeled by the Euler-Bernoulli beam theory.

A further refined model has been given by Chasalevris et al. [12] to predict the dynamic behaviour of a cracked beam with two transverse surface cracks. Each crack is characterized by its depth, position and relative angle. The compliance matrix is calculated at any angle of rotation. Further, an improved analytical method for calculating the natural frequencies of a uniform beam with a large number of cracks is proposed by Shifrin and Ruotolo [13] & Kisa and Brandon [14]. Similar analysis has been done using modified Fourier series by Zheng and Fan [15] for a non uniform beam having varying cross section. Khiemand and Lien [16] proposed a more simplified method for evaluating the natural frequencies of beams with an arbitrary number of cracks based on the use of the rotational spring model of cracks. Kisa [17] has presented a new method for the numerical modelling of the free vibration of a cantilever composite beam having multiple open and non-propagating cracks. The method integrates the fracture mechanics and the joint interface mechanics to couple substructures. The finite element and the component mode synthesis methods are used to model the problem.

2. Theoretical Analysis

2.1 Local flexibility of a cracked beam under bending and axial loading

The presence of a transverse surface crack of depth 'a_1'and 'a_2' on beam of width 'B' and height 'W' introduces a local flexibility, which can be defined in matrix form, the dimension of which depends on the degrees of freedom. Here a 2×2 matrix is considered. A cantilever beam is subjected to axial force (P_1) and bending moment (P_2), shown in Fig.1 (a), which gives coupling with the longitudinal and transverse motion. The cross sectional view of the beam is shown in Fig.1 (b).

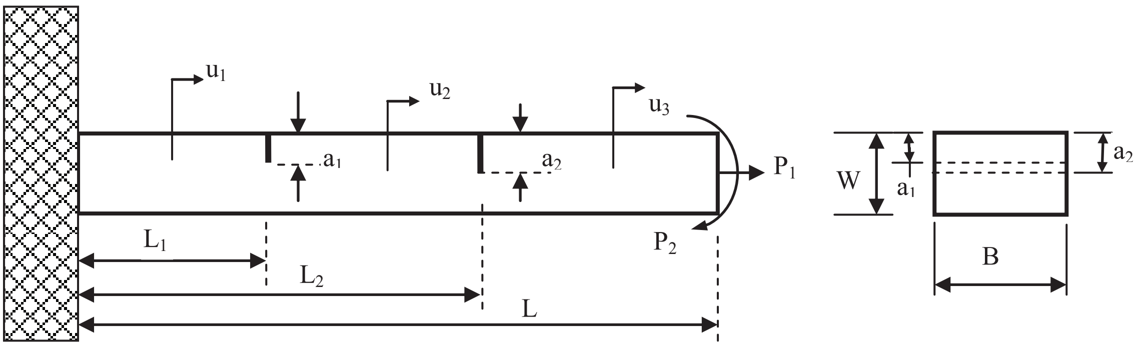

Fig.1. Geometry of beam, (a) Cantilever beam with double cracks; (b) Cross-sectional view of the beam

The strain energy release rate at the fractured section can be written as [18];

$$J = \frac{1}{E'}\left(K_{I1} + K_{I2}\right)^2, \tag{1}$$

Where $\dfrac{1}{E'} = \dfrac{1-v^2}{E}$ (for plane strain condition);

$\dfrac{1}{E'} = \dfrac{1}{E}$ (for plane stress condition)

K_{I1}, K_{I2} are the stress intensity factors of mode I (opening of the crack) for load P_1 and P_2 respectively. The values of stress intensity factors from earlier studies [18] are :

$$K_{I1} = \frac{P_1}{BW}\sqrt{\pi a}\left(F_1\left(\frac{a}{W}\right)\right),\ K_{I2} = \frac{6P_2}{BW^2}\sqrt{\pi a}\left(F_2\left(\frac{a}{W}\right)\right)$$

Where expressions for F_1 and F_2 are as follows

$$F_1\left(\frac{a}{W}\right) = \left(\frac{2W}{\pi a}\tan\left(\frac{\pi a}{2W}\right)\right)^{0.5}\left\{\frac{0.752 + 2.02(a/W) + 0.37\ \ 1-\sin(\pi a/2W\ ^{3}}{\cos\ \pi a/2W}\right\} \tag{2}$$

$$F_2\left(\frac{a}{W}\right) = \left(\frac{2W}{\pi a}\tan\left(\frac{\pi a}{2W}\right)\right)^{0.5}\left\{\frac{0.923 + 0.199\ \ 1-\sin(\pi a/2W\ ^{4}}{\cos\ \pi a/2W}\right\} \tag{3}$$

The expressions for F_1 (a/W) and F_2 (a/W) are the functions used for calculation of the stress intensity factors K_{11} and K_{12}. Let U_t be the strain energy due to the crack. Then from Castiglione's theorem, the additional displacement along the force P_i is:

$$u_i = \frac{\partial U_t}{\partial P_i} \tag{4}$$

The strain energy will have the form, $U_t = \int_0^{a_1}\frac{\partial U_t}{\partial a}da = \int_0^{a_1} J da$ (5)

Where $J = \dfrac{\partial U_t}{\partial a}$ the strain energy density function.

From Equation (1) and Equation (2), thus we have

$$u_i = \frac{\partial}{\partial P_i}\left[\int_0^{a_1} J(a)da\right] \tag{6}$$

The flexibility influence co-efficient C_{ij} will be, by definition

$$C_{ij} = \frac{\partial u_i}{\partial P_j} = \frac{\partial^2}{\partial P_i \partial P_j}\int_0^{a_1} J(a)da \tag{7}$$

and can be written in terms of ξ =a/W

$$C_{ij} = \frac{BW}{E'}\frac{\partial^2}{\partial P_i \partial P_j}\int_0^{\xi_1}(K_{I1} + K_{I2})^2 d\xi \tag{8}$$

From Equation (7), calculating C_{11}, C_{12} (= C_{21}) and C_{22} we get

$$\overline{C_{11}} = C_{11}\frac{BE'}{2\pi};\ \overline{C_{12}} = C_{12}\frac{E'BW}{12\pi} = \overline{C_{21}};\ \overline{C_{22}} = C_{22}\frac{E'BW^2}{72\pi}$$

The local stiffness matrix can be obtained by taking the inversion of compliance matrix i.e.

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^{-1}$$

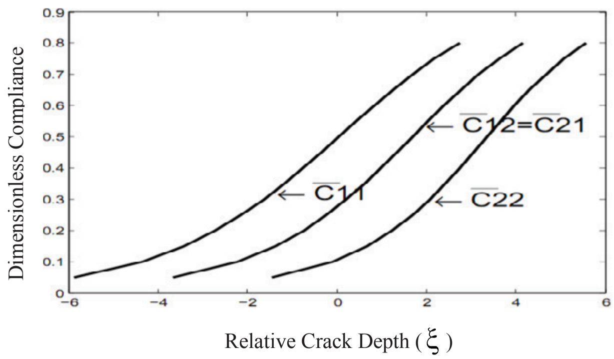

Fig.2. Relative Crack Depth (ξ) vs. Dimensionless Compliance (ln $(\overline{C}_{i-1,2,j=1,2})$)

2.2 Analysis of vibration characteristics of the cracked beam

Taking $u_1(x, t)$, $u_2(x,t)$, $u_3(x,t)$ as the amplitudes of longitudinal vibration for the sections before, in-between and after the crack and $y_1(x, t)$, $y_2(x, t)$, $y_3(x, t)$ are the amplitudes of bending vibration for the same sections.

The normal function for the system can be defined as

$$\bar{y}_1(\bar{x}) = A_1\cosh(\bar{K}_y\bar{x}) + A_2\sinh(\bar{K}_y\bar{x}) + A_3\cos(\bar{K}_y\bar{x}) + A_4\sin(\bar{K}_y\bar{x}) \quad (9)$$

$$\bar{y}_2(\bar{x}) = A_5\cosh(\bar{K}_y\bar{x}) + A_6\sinh(\bar{K}_y\bar{x}) + A_7\cos(\bar{K}_y\bar{x}) + A_8\sin(\bar{K}_y\bar{x}) \quad (10)$$

$$\bar{y}_3(\bar{x}) = A_9\cosh(\bar{K}_y\bar{x}) + A_{10}\sinh(\bar{K}_y\bar{x}) + A_{11}\cos(\bar{K}_y\bar{x}) + A_{12}\sin(\bar{K}_y\bar{x}) \quad (11)$$

$$\bar{u}_1(\bar{x}) = A_{13}\cos(\bar{K}_u\bar{x}) + A_{14}\sin(\bar{K}_u\bar{x}) \quad (12)$$

$$\bar{u}_2(\bar{x}) = A_{15}\cos(\bar{K}_u\bar{x}) + A_{16}\sin(\bar{K}_u\bar{x}) \quad (13)$$

$$\bar{u}_3(\bar{x}) = A_{17}\cos(\bar{K}_u\bar{x}) + A_{18}\sin(\bar{K}_u\bar{x}) \quad (14)$$

Where $\bar{x} = \frac{x}{L}, \bar{u} = \frac{u}{L}, \bar{y} = \frac{y}{L}, \beta_1 = \frac{L_1}{L},\ \beta_2 = \frac{L_2}{L}$

$$\bar{K}_u = \frac{\omega L}{C_u}, C_u = \left(\frac{E}{\rho}\right)^{1/2}, \bar{K}_y = \left(\frac{\omega L^2}{C_y}\right)^{1/2}, C_y = \left(\frac{EI}{\mu}\right)^{1/2}, \mu = A\rho$$

A_i, (I = 1, 18) Constants are to be determined, from boundary conditions. The boundary conditions of the cantilever beam in consideration are:

$$\bar{u}_1(0) = 0; \quad \bar{y}_1(0) = 0; \quad \bar{y}_1'(0) = 0; \quad \bar{u}_3'(1) = 0; \quad \bar{y}_3''(1) = 0; \quad \bar{y}_3'''(1) = 0$$

At the cracked section:

$$\bar{u}_1'(\beta_1) = \bar{u}_2'(\beta_1); \quad \bar{y}_1(\beta_1) = \bar{y}_2(\beta_1); \quad \bar{y}_1''(\beta_1) = \bar{y}_2''(\beta_1); \quad \bar{y}_1'''(\beta_1) = \bar{y}_2'''(\beta_1)$$

$$\bar{u}_2'(\beta_2) = \bar{u}_3'(\beta_2); \quad \bar{y}_2(\beta_2) = \bar{y}_3(\beta_2); \quad \bar{y}_2''(\beta_2) = \bar{y}_3''(\beta_2); \quad \bar{y}_2'''(\beta_2) = \bar{y}_3'''(\beta_2)$$

Also at the cracked section L_1, we have :

$$AE\frac{du_1(L_1)}{dx} = K_{11}(u_2(L_1) - u_1(L_1)) + K_{12}\left(\frac{dy_2(L_1)}{dx} - \frac{dy_1(L_1)}{dx}\right)$$

[Due to the discontinuity of axial deformation to the left and right of the crack, the boundary conditions given in Equation (15) arise.]

Multiplying both sides of the equation (15) by $\frac{AE}{LK_{11}K_{12}}$ we get;

$$M_1M_2\bar{u}_1'(\beta_1) = M_2(\bar{u}_2(\beta_1) - \bar{u}_1(\beta_1) + M_1(\bar{y}_2'(\beta_1) - \bar{y}_1'(\beta_1)) \tag{15}$$

Similarly, $EI\frac{d^2y_1(L_1)}{dx^2} = K_{21}(u_2(L_1) - u_1(l_1)) + K_{22}\left(\frac{dy_2(L_1)}{dx} - \frac{dy_1(L_1)}{dx}\right)$

Due to the discontinuity of slope to the left and right of the crack the boundary conditions given in equation (16) arises. Multiplying both sides of the Equation (16) by $\frac{EI}{L^2K_{22}K_{21}}$ we get,

$$M_3M_4\bar{y}_1''(\beta_1) = M_3(\bar{u}_2(\beta_1) - \bar{u}_1(\beta_1)) + M_4(\bar{y}_2'(\beta_1) - \bar{y}_1'(\beta)) \tag{16}$$

Where, $M_1 = \frac{AE}{LK_{11}}, M_2 = \frac{AE}{K_{12}}, M_3 = \frac{EI}{LK_{22}}, M_4 = \frac{EI}{L^2K_{21}}$

Similarly at the crack section L_2 we can have the expression;

$$M_5M_6\bar{u}_2'(\beta_2) = M_6 \quad \bar{u}_3(\beta_2) - \bar{u}_2(\beta_2) \quad + M_5(\bar{y}_3'(\beta_2) - \bar{y}_2'(\beta_2)) \tag{17}$$

$$M_7M_8\bar{y}_2''(\beta_2) = M_7 \quad \bar{u}_3(\beta_3) - \bar{u}_2(\beta_2) \quad + M_8(\bar{y}_3'(\beta_2) - \bar{y}_2'(\beta_2)) \tag{18}$$

Where $M_5 = \frac{AE}{LK_{11}}, M_6 = \frac{AE}{K_{12}}, M_7 = \frac{EI}{LK_{22}}, M_8 = \frac{EI}{L^2K_{21}}$

The normal functions, Equation (9) to Equation (14) along with the boundary conditions as mentioned above, yield the characteristic equation of the system as: $|Q| = 0$

This determinant is a function of natural circular frequency (ω), the relative locations of the crack (β_1, β_2) and the local stiffness matrix (K) which in turn is a function of the relative crack depth ($\xi_1 = a_1/W$, $\xi_2 = a_2/W$). Matrix Q and its elements are given explicitly in Appendix A.

The results of the theoretical analysis for the first three mode shapes for un-cracked and cracked beam are shown in figure 4, 5 and 6.

3. Results and Discussions

Beam specification

In the current investigation using theoretical analysis of a cracked cantilever beam, the following dimensions are being considered.

1) Length of the Beam, 'L' = 800 mm.
2) Width of the Beam, 'W' = 50 mm.
3) Height of the Beam, 'B' = 6 mm.
4) The relative crack depth ($\xi_1 = a_1 / W$) = varies from 0.05 to 0.8
5) The relative crack depth ($\xi_2 = a_2 / W$) = varies from 0.05 to 0.8
5) The relative crack location ($\beta_1 = L_1/L$) = varies from 0.0125 to 0.65
6) The relative crack location ($\beta_2 = L_2/L$) = varies from 0.125 to 0.875

Based on the results obtained from the numerical method analyses, the following discussions can be made. The dimensionless compliances ($\bar{C}_{11}$, $\bar{C}_{12} = \bar{C}_{21}$, $\bar{C}_{22}$) increase with the increase in relative crack depth as shown in Fig. 2.

Based on the results obtained from the theoretical analyses for a multi-cracked cantilever beam, the following discussions can be made. From Fig 3a. 3b. 3c. it is seen that the relative frequency for mode-I is on increasing trend while for other two modes attains minimality and maximality at different crack location. Moreover, it is observed that these variations of the relative frequencies are showing a steeper change with the increase of crack depth. A sharp decline in the relative natural frequencies for the first mode has been observed after a certain crack depth (Fig 3d.).

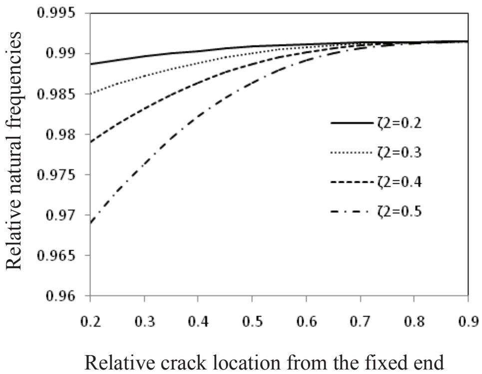

Fig. 3a. Relative natural frequencies vs. Relative crack location from the fixed end for mode-I vibration

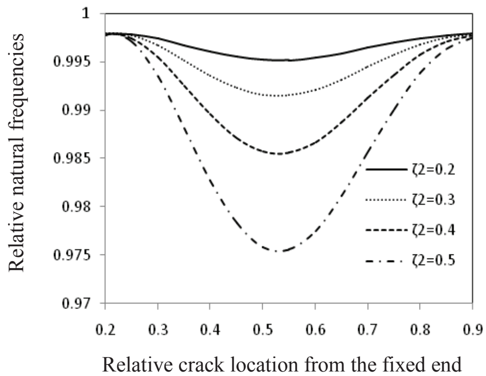

Fig. 3b. Relative natural frequencies vs. Relative crack location from the fixed end for mode-II vibration

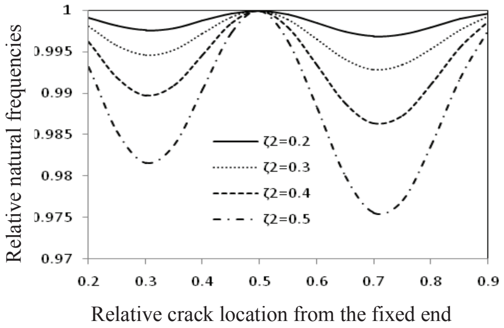

Fig. 3c. Relative natural frequencies vs. Relative crack location from the fixed end for mode-III vibration

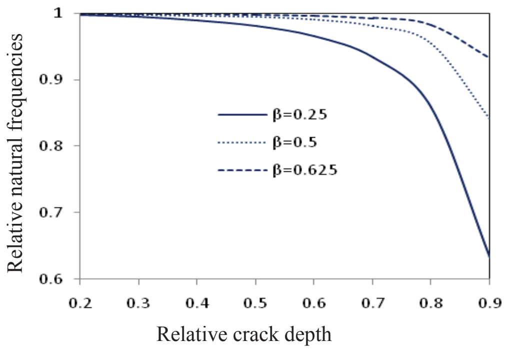

Fig. 3d. Relative natural frequencies vs. Relative crack depth for mode-I vibration

It is observed through the magnified views at the crack locations (with $\beta_1 = 0.125$ and $\beta_2 = 0.25$) that there are reasonable changes in mode shapes due to the presence of crack with higher intensity in the beam (Fig 4., Fig 5. and Fig 6.). Moreover, these changes in mode shapes are more prominent at the second crack position and for the higher mode of vibration.

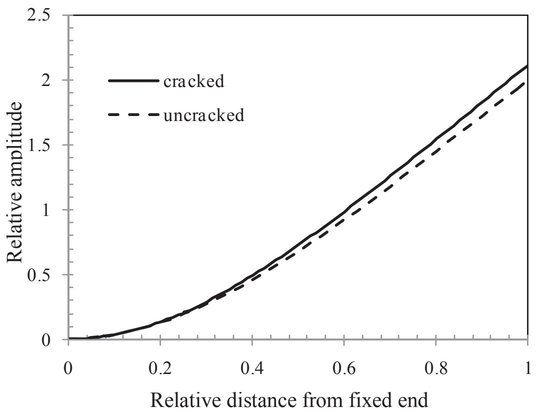

Fig. 4a. First mode vibration with $\beta_{1=0.125}$ and $\beta_{2=0.25}$

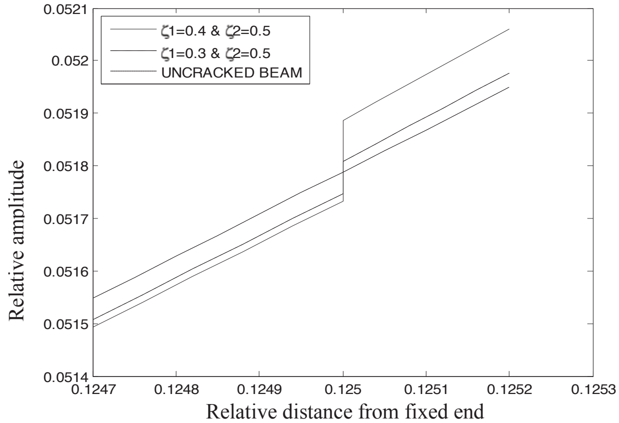

Fig. 4b. Magnified view at the first crack location

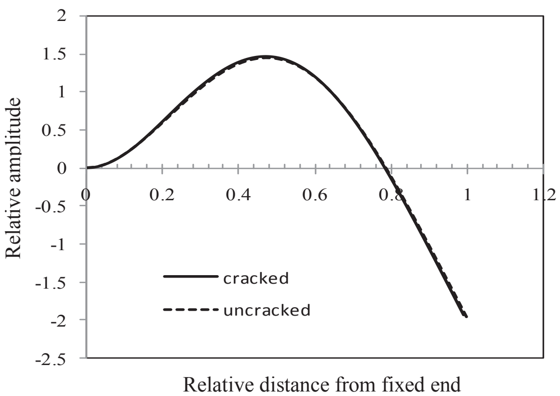

Fig. 5a. Second mode vibration with $\beta_{1=0.125}$ and $\beta_{2=0.25}$

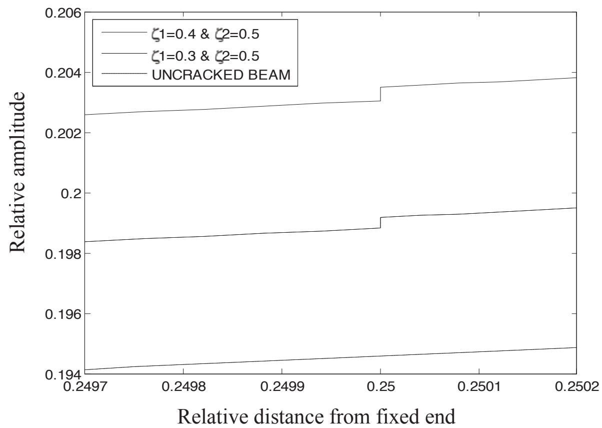

Fig. 4c. Magnified view at the second crack location

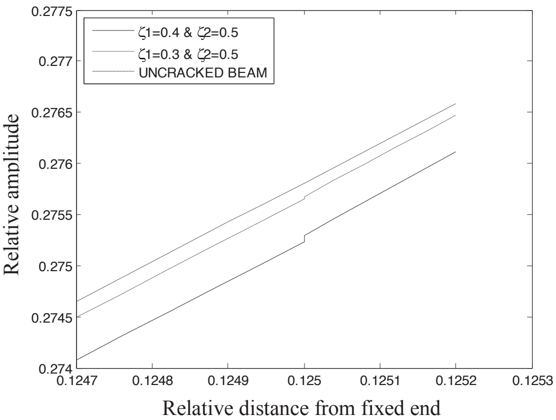

Fig. 5b. Magnified view at the first crack location

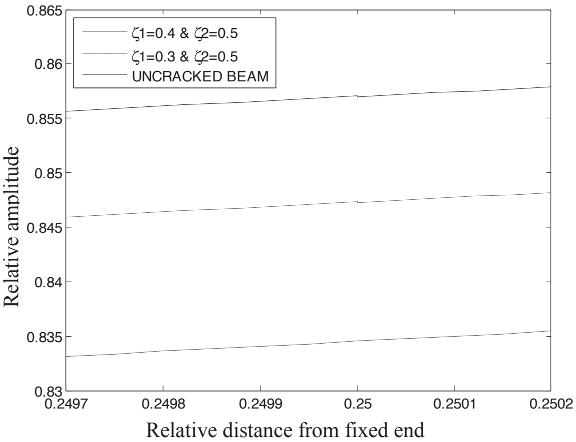

Fig. 5c. Magnified view at the second crack location

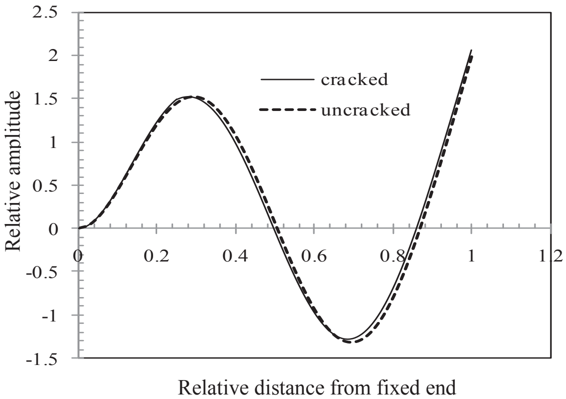

Fig. 6a. Third mode vibration with $\beta_{1}=0.125$ and $\beta_{2}=0.25$

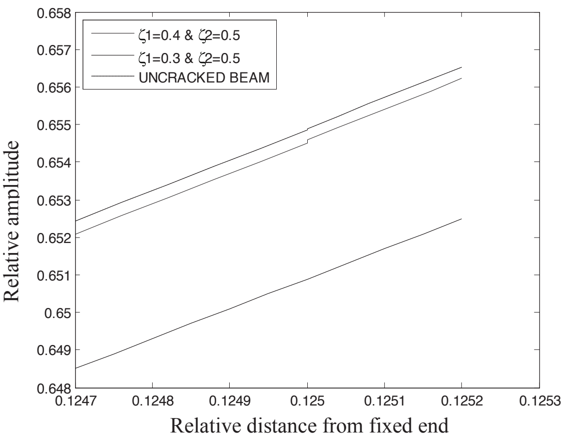

Fig. 6b. Magnified view at the first crack location

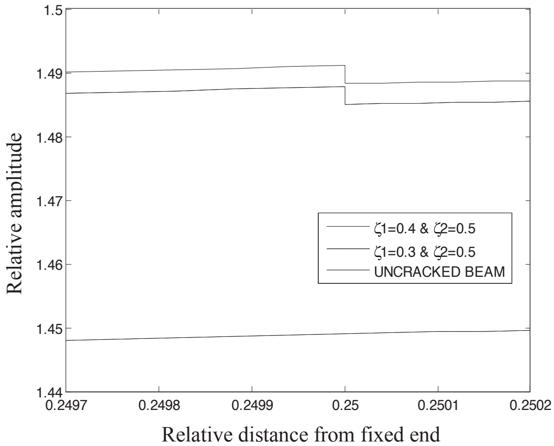

Fig. 6c. Magnified view at the second crack location

A program is written in MATLAB for the computation of the lowest three natural frequencies and their respective relative amplitudes of a cantilever beam with double cracks for various crack sizes and crack locations. The computed values of such frequencies are given in Table 1. Young's modulus and the density of the beam are E= 70GPa and ρ= 2720 kg/m^3, respectively, throughout this study.

Table 1. Natural frequencies of a beam with double cracks

β_1	β_2	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)
ξ_1 =0.2, ξ_2 =0.3				
0.125	0.25	47.8359	302.176	843.492
	0.5	48.025	300.333	847.038
	0.75	48.088	301.587	841.515
0.25	0.375	48.0139	301.415	842.306
	0.625	48.1339	300.874	841.523
	0.875	48.159	302.414	844.384
0.375	0.5	48.1389	300.151	845.626
	0.625	48.1839	300.446	841.625
	0.875	48.208	301.977	844.504
0.625	0.75	48.256	301.157	839.856
	0.875	48.261	301.742	844.196
ξ_1 =0., ξ_2 =0.4				
0.125	0.25	47.596	302.11	840.207
	0.5	47.958	298.558	847.035
	0.75	48.082	300.961	836.371
0.25	0.375	47.8769	300.417	839.372
	0.625	48.11	299.366	837.875
	0.875	48.159	302.346	843.316
0.375	0.5	48.074	298.39	845.62
	0.625	48.1599	298.947	837.955
	0.875	48.208	301.909	843.429
0.625	0.75	48.25	300.535	834.78
	0.875	48.261	301.674	843.134
ξ_1 =0.3, ξ_2 =0.5				
0.125	0.25	46.969	301.641	834.681
	0.5	47.6189	295.219	847.014
	0.75	47.8419	299.539	827.676
0.25	0.375	47.499	298.703	832.733
	0.625	47.923	296.772	829.855
	0.875	48.011	302.188	839.495
0.375	0.5	47.8779	294.886	843.775
	0.625	48.0349	295.846	829.961
	0.75	48.124	301.187	839.739

0.625	0.75	48.227	298.566	824.294
	0.875	48.245	300.646	839.114
ξ_1 =0.4, ξ_2 =0.5				
0.125	0.25	46.61	301.061	834.633
	0.5	47.2449	294.644	846.99
	0.75	47.462	298.982	827.643
0.25	0.375	47.2639	298.654	829.809
	0.625	47.6809	296.721	826.593
	0.875	47.768	302.125	836.278
0.375	0.5	47.742	293.988	840.778
	0.625	47.897	294.934	826.824
	0.75	47.986	300.192	836.774
0.625	0.75	48.2019	297.085	821.157
	0.875	48.222	299.138	835.577

4. Conclusion

The conclusions drawn from the above discussion shows that the mode shapes and natural frequencies of the cracked elastic structures are strongly influenced by the crack location and its intensity. The significant changes in mode shapes are observed at the vicinity of crack location. The positions of the cracks in relation to each other affect significantly the changes in the frequencies of the natural vibrations in the case of an equal relative depth of the cracks. Any decrease in the natural frequency is largest if the cracks are near to each other; when the distance between the cracks increases the frequencies of the beam natural vibrations also tend to the natural vibration frequencies of a system with a single crack. In the case of two cracks of different depths, the larger crack has the most significant effect on the natural vibration frequencies. This is evident for the first natural vibration of a cantilever beam. For other modes of vibration this is not so clear, because the influence of a crack location at a node is negligible. These changes in mode shapes and natural frequencies will be helpful in prediction of crack location and its intensity.

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Appendix A

Elements of the Q-Matrix

$G_1=\cosh(\overline{K}_y\beta_1)$, $G_2=\sinh(\overline{K}_y\beta_1)$, $G_3=\cosh(\overline{K}_y)$, $G_4=\sinh(\overline{K}_y)$, $G_5=\cos(\overline{K}_y\beta_1)$, $G_6=\sin(\overline{K}_y\beta_1)$,

$G_7=\cos(\overline{K}_y)$, $G_8 = \sin(\overline{K}_y)$, $G_9=\cosh(\overline{K}_y\beta_2)$, $G_{10}=\sinh(\overline{K}_y\beta_2)$, $G_{11}=\cos(\overline{K}_y\beta_2)$,

$G_{12}=\sin(\overline{K}_y\beta_2)$, $T_5=\cos(\overline{K}_u\beta_1)$, $T_6=\sin(\overline{K}_u\beta_1)$, $T_7=\cos(\overline{K}_u)$, $T_8=\sin(\overline{K}_u)$, $T_9=\cos(\overline{K}_u\beta_2)$,

$T_{10}=\sin(\overline{K}_u\beta_2)$

$M_{12}=\frac{M_1}{M_2}$, $M_{34}=\frac{M_3}{M_4}$

$S_1= G_2+ M_3\overline{K}_yG_1$; $S_2=G_1+M_3\overline{K}_yG_2$; $S_3=-G_6-M_3\overline{K}_yG_5$; $S_4=G_5-M_3\overline{K}_yG_6$; $S_5=\frac{M_{34}}{\overline{K}_y}T_5$;

$S_6=\frac{M_{34}}{\bar{K}_y}T_6$; $S_7=-\frac{M_{34}}{\bar{K}_y}T_5$; $S_8=-\frac{M_{34}}{\bar{K}_y}T_6$, $M_{56}=\frac{M_5}{M_6}$; $M_{78}=\frac{M_7}{M_8}$;

$SS_1=G_{10}+M_7\bar{K}_yG_9$; $SS_2=G_9+M_7\bar{K}_yG_{10}$; $SS_3=-G_{12}-M_7\bar{K}_yG_{11}$; $SS_4=G_{11}-M_7\bar{K}_yG_{12}$;

$SS_5=\frac{M_{78}}{\bar{K}_y}T_9$; $SS_6=\frac{M_{78}}{\bar{K}_y}T_{10}$; $SS_7=-\frac{M_{78}}{\bar{K}_y}T_9$; $SS_8=-\frac{M_{78}}{\bar{K}_y}T_{10}$, $S_9=M_{12}\bar{K}_yG_2$;

$S_{10}=M_{12}\bar{K}_yG_1$; $S_{11}=-M_{12}\bar{K}_yG_6$; $S_{12}=M_{12}\bar{K}_yG_5$; $S_{13}=-M_{12}\bar{K}_yG_2$; $S_{14}=-M_{12}\bar{K}_yG_1$;

$S_{15}=M_{12}\bar{K}_yG_6$

$S_{16}=-M_{12}\bar{K}_yG_5$; $S_{17}=T_5-M_1\bar{K}_uT_6$; $S_{18}=T_6+M_1\bar{K}_uT_5$; $SS_9=M_{56}\bar{K}_yG_{10}$; $SS_{10}=M_{56}\bar{K}_yG_9$;

$SS_{11}=-M_{56}\bar{K}_yG_{12}$; $SS_{12}=M_{56}\bar{K}_yG_{11}$; $SS_{13}=-M_{56}\bar{K}_yG_{10}$; $SS_{14}=-M_{56}\bar{K}_yG_9$;

$SS_{15}=M_{56}\bar{K}_yG_{12}$;

$SS_{16}=-M_{56}\bar{K}_yG_{11}$; $SS_{17}=T_9-M_5\bar{K}_uT_{10}$; $SS_{18}=T_{10}+M_5\bar{K}_uT_9$;

Q – Matrix

$$
\begin{vmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_3 & G_4 & -G_7 & -G_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G_4 & G_3 & G_8 & -G_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
G_1 & G_2 & -G_5 & -G_6 & -G_1 & -G_2 & G_5 & G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
G_2 & G_1 & G_6 & -G_5 & -G_2 & -G_1 & -G_6 & G_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
G_1 & G_2 & G_5 & G_6 & -G_1 & -G_2 & -G_5 & -G_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & G_9 & G_{10} & -G_{11} & -G_{12} & -G_9 & -G_{10} & G_{11} & G_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & G_{10} & G_9 & G_{12} & -G_{11} & -G_{10} & -G_9 & -G_{12} & G_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & G_9 & G_{10} & G_{11} & G_{12} & -G_9 & -G_{10} & -G_{11} & -G_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
S_1 & S_2 & S_3 & S_4 & -G_2 & -G_1 & G_6 & -G_5 & 0 & 0 & 0 & 0 & S_5 & S_6 & S_7 & S_8 & 0 & 0 \\
0 & 0 & 0 & 0 & SS_1 & SS_2 & SS_3 & SS_4 & -G_{10} & -G_9 & G_{12} & -G_{11} & 0 & 0 & SS_5 & SS_6 & SS_7 & SS_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_8 & T_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_6 & T_5 & T_6 & -T_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_{10} & T_9 & T_{10} & -T_9 \\
S_9 & S_{10} & S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & 0 & 0 & 0 & 0 & S_{17} & S_{18} & -T_5 & -T_6 & 0 & 0 \\
0 & 0 & 0 & 0 & SS_9 & SS_{10} & SS_{11} & SS_{12} & SS_{13} & SS_{14} & SS_{15} & SS_{16} & 0 & 0 & SS_{17} & SS_{18} & -T_9 & -T_{10}
\end{vmatrix}
$$